The blowup algebra

Def: R a ring, $I \subseteq R$ an ideal, then the <u>blowup algebra</u> (or, more commonly, <u>Rees algebra</u>) of I in R is the R-algebra

$$B_I R := R \oplus I \oplus I^2 \oplus \dots .$$

Often to make the indexing easier, we add in a variable, so we write

$$f = a_0 + a_1 t + a_2 t^2 + \dots \in R[tI] \cong B_I R.$$

In this way, $B_I R$ becomes a subring of $R[t]$.

Note: $B_I R / I B_I R = R/I \oplus I/I^2 \oplus \dots = gr_I R.$

Ex: Let $R = k[x_1, x_2]$, $I = (x_1, x_2)$. Then there's a natural map

$$\begin{aligned} k[x_1, x_2, \overbrace{y_1, y_2}^{\text{homog. coords}}] &\longrightarrow k[x_1, y_1, t] \\ x_i &\longmapsto x_i \\ y_i &\longmapsto x_i t \end{aligned}$$

The image is elements of the form $a_0 + a_1 t + a_2 t^2 + \dots$ where $a_i \in I^i$, so the image is exactly $B_I R$.

Note that $x_1 y_2 \longmapsto x_1 x_2 t$ and $x_2 y_1 \mapsto x_1 x_2 t$. In fact, the kernel is $(x_1 y_2 - x_2 y_1)$, and the corresponding algebraic subset Z is the Blowup of $\mathbb{A}^2$ at the origin.

Note that $R \hookrightarrow k[x_1, x_2, \overset{\text{corr. to } \mathbb{A}^2 \times \mathbb{P}^1}{y_1, y_2]} \longrightarrow \underbrace{k[x_1,x_2,y_1,y_2]/(x_1y_2 - x_2y_1)}_{\cong B_I R}$

so we get a map $Z \to \mathbb{A}^2$.

The y_i in $k[x_1,x_2,y_1,y_2]$ are homogeneous, which mean the points corr. to ideals of the form $(x_1 - a_1, x_2 - a_2, b_1y_1 - b_2y_2)$

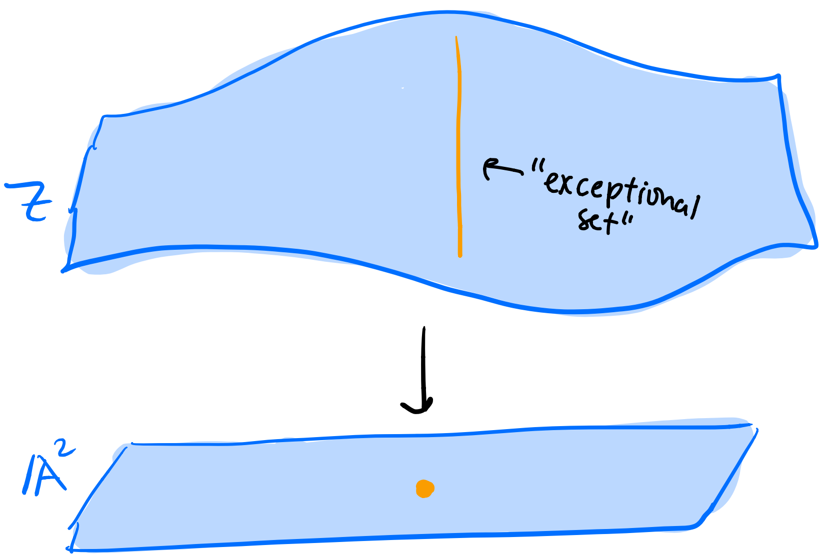
not both 0

So for $(a_1, a_2) \in \mathbb{A}^2$, not the origin, the point in Z lying over it corr. to $(x_1 - a_1, x_2 - a_2, a_2y_1 - a_1y_2)$.

Over $(0,0)$, we have $(x_1, x_2, b_1y_1 - b_2y_2)$. This gives us

b_1, b_2 not both 0

one dim. worth of freedom. In fact it's a $\mathbb{P}^1$!



The blowup of $\mathbb{A}^2$ at $(0,0)$ is $\mathbb{A}^2$ w/ the origin replaced by a projective line.

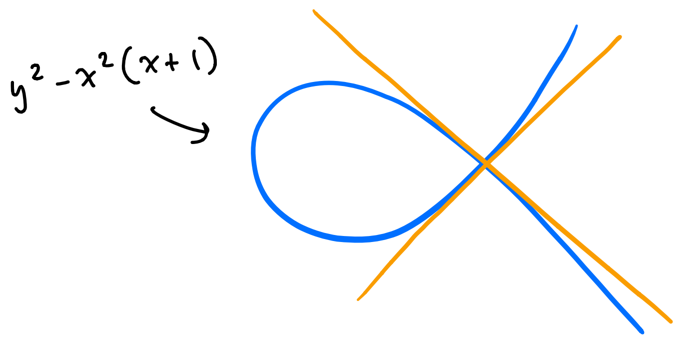
The exceptional set of the blowup is the preimage of the origin, corresponding to the ring $B_IR / IB_IR = gr_I R$.

Each point in the exceptional set corresponds to a normal direction at the origin.

More generally, if $R = k[x_1, \dots, x_n]/J$ and $I = (x_1, \dots, x_n)$ s.t. $J \subseteq I$, define $X = V(J) \subseteq \mathbb{A}^n$. Then $\underset{(x_1,\dots,x_n)}{\overset{\uparrow}{0}} \in X$.

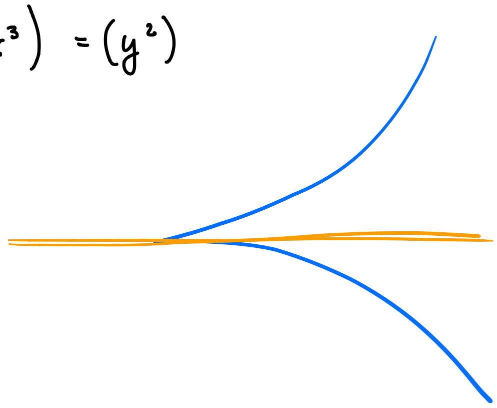
The <u>tangent cone</u> corr. to $\mathrm{in}_I(J) \subseteq k[x_1, \dots, x_n]$, i.e. it is $\mathrm{Spec}\left(\mathrm{gr}_I R / \mathrm{in}_I J\right)$ and $\mathrm{gr}_I R / \mathrm{in}\, J \underset{\text{see HW}}{\overset{\uparrow}{=}} \mathrm{gr}_I (R/J)$. It consists of the limits of secant lines through the origin.

Ex: $\mathrm{in}\left(y^2 - x^2(x+1)\right) = (y^2 - x^2) = (y-x)(y+x)$



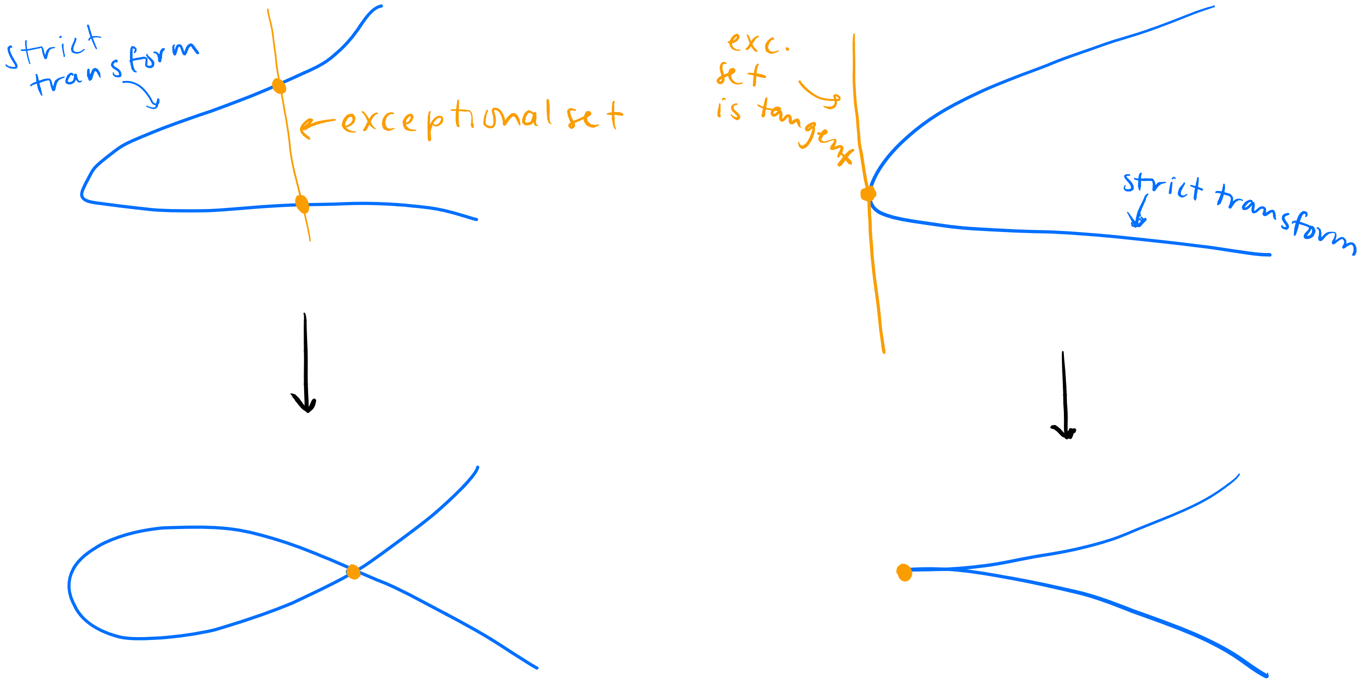
tangent cone = $\mathrm{Spec}\left(k[x,y]/(y-x)(y+x)\right)$

Ex: $\mathrm{in}(y^2 - x^3) = (y^2)$



tangent cone is $\mathrm{Spec}\left(k[x,y]/(y^2)\right)$

When blowing up the plane at the origin, each line in the tangent cone corresponds to a point in the fiber over the origin in the preimage of the curve


strict transform
exceptional set
exc. set is tangent
strict transform